THE GROTHENDIECK TOPOS OF GENERALIZED FUNCTIONS

PAOLO GIORDANO

We present a new approach to generalized functions, so-called generalized smooth functions (GSF). GSF are set-theoretical maps defined on, and taking values in the non-Archimedean ring (i.e. with real, infinite and infinitesimal numbers) of Robinson-Colombeau, and form a concrete category which unifies and extends Schwartz distributions and Colombeau generalized functions. The calculus of these generalized functions is closely related to classical analysis, with point values, the usual rules for differentiation and integration, free composition and hence non linear operations. We have classical theorems such as: intermediate value theorem, mean value theorems, extreme value theorem, several forms of Taylor formula, local and global inverse and implicit function theorems, generalized sheaf property in the sharp topology; Multidimensional integration, monotone and dominated convergence theorems; A full theory of non-Archimedean locally convex topological vector spaces of GSF; A theory of singular nonlinear ODE with Banach fixed point theorem, Picard-Lindelöf theorem, maximal set of existence, Gronwall theorem, flux properties, continuous dependence on initial conditions, full compatibility with classical smooth solutions; Calculus of variations with: fundamental lemma, second variation and minimizers, necessary Legendre condition, Jacobi fields, conjugate points and Jacobi's theorem, Noether's theorem. Using GSF, we can also prove a Picard-Lindelöf theorem for nonlinear singular PDE in normal form. Statements of these results are faithful transfers of the corresponding classical results. Finally, we can define a concrete site and hence a Grothendieck topos of sheaves of generalized functions which contains the sheaves of Schwartz distributions and Colombeau generalized functions. This universe (topos) faithfully embed classical smooth manifolds and is closed with respect to products, sums, pullbacks, push-outs, equalizers, infinite-dimensional function spaces, arbitrary subspaces, etc. In this framework we can hence consider a generalization of Souriau diffeological spaces to generalized functions. We hence present the planned future developments of this theory concerning differential geometry of these generalized diffeological spaces and the related Hamiltonian mechanics.

References

[1] A. Lecke, L. Luperi Baglini, P. Giordano, *The classical theory of calculus of variations for generalized functions.* Accepted in Advances in Nonlinear Analysis. DOI: 10.1515/anona-2017-0150.

[2] P. Giordano, M. Kunzinger, *Inverse Function Theorems for Generalized Smooth Functions.* Chapter in "Generalized Functions and Fourier Analysis", Volume **260** of the series Operator Theory: Advances and Applications pp. 95-114.

[3] P. Giordano, M. Kunzinger, H. Vernaeve, *Strongly internal sets and generalized smooth functions.* Journal of Mathematical Analysis and Applications, volume **422**, issue 1, 2015, pp. 56-71.

[4] P. Giordano, M. Kunzinger, *A convenient notion of compact set for generalized functions.* Proceedings of the Edinburgh Mathematical Society, Volume 61, Issue 1, February 2018, pp. 57-92.

[5] P. Giordano, home page: www.mat.univie.ac.at/~giordap7/

Wolfgang Pauli Institute, Vienna
Email address: paolo.giordano@univie.ac.at